Lecture 9

Waves in Gyrotropic Media, Polarization

9.1 Gyrotropic Media

This section presents deriving the permittivity tensor of a gyrotropic medium in the ionsphere. Our ionosphere is always biased by a static magnetic field due to the Earth's magnetic field [68]. But in this derivation, to capture the salient feature of the physics with a simple model, one assumes that the ionosphere has a static magnetic field polarized in the z direction, namely that $\mathbf{B} = \hat{z}B_0$. Now, the equation of motion from the Lorentz force law for an electron with $q = -e$, in accordance with Newton's law, becomes

$$m_e \frac{d\mathbf{v}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{9.1.1}$$

Next, let us assume that the electric field is polarized in the xy plane. The derivative of $\mathbf{v}$ is the acceleration of the electron, and also, $\mathbf{v} = d\mathbf{r}/dt$ where $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$. And in the frequency domain, the above equation in the cartesian coordinates becomes

$$m_e\omega^2 x = e(E_x + j\omega B_0 y) \tag{9.1.2}$$

$$m_e\omega^2 y = e(E_y - j\omega B_0 x) \tag{9.1.3}$$

The above equations cannot be solved easily for x and y in terms of the electric field because they correspond to a two-by-two matrix system with cross coupling between the unknowns x and y. But they can be simplified as follows: We can multiply (9.1.3) by $\pm j$ and add it to (9.1.2) to get two decoupled equations [69]:

$$m_e\omega^2(x + jy) = e[(E_x + jE_y) + \omega B_0(x + jy)] \tag{9.1.4}$$

$$m_e\omega^2(x - jy) = e[(E_x - jE_y) - \omega B_0(x - jy)] \tag{9.1.5}$$

Defining new variables such that

$$s_\pm = x \pm jy \tag{9.1.6}$$

$$E_\pm = E_x \pm jE_y \tag{9.1.7}$$

then (9.1.4) and (9.1.5) become

$$m_e\omega^2 s_\pm = e(E_\pm \pm \omega B_0 s_\pm) \tag{9.1.8}$$

Thus, solving the above yields

$$s_\pm = \frac{e}{m_e\omega^2 \mp eB_0\omega} E_\pm = C_\pm E_\pm \tag{9.1.9}$$

where

$$C_\pm = \frac{e}{m_e\omega^2 \mp eB_0\omega} \tag{9.1.10}$$

By this manipulation, the above equations (9.1.2) and (9.1.3) transform to new equations where there is no cross coupling between $s_\pm$ and $E_\pm$. The mathematical parlance for this is the diagnolization of a matrix equation [70]. Thus, the new equation can be solved easily.

Next, one can define $P_x = -Nex$, $P_y = -Ney$, and that $P_\pm = P_x \pm jP_y = -Nes_\pm$. Then it can be shown that

$$P_\pm = \varepsilon_0 \chi_\pm E_\pm \tag{9.1.11}$$

The expression for $\chi_\pm$ can be derived, and they are given as

$$\chi_\pm = -\frac{NeC_\pm}{\varepsilon_0} = -\frac{Ne}{\varepsilon_0}\frac{e}{m_e\omega^2 \mp eB_o\omega} = -\frac{{\omega_p}^2}{\omega^2 \mp \Omega\omega} \tag{9.1.12}$$

where Ω and ω_p are the cyclotron frequency[1] and plasma frequency, respectively.

$$\Omega = \frac{eB_0}{m_e}, \quad {\omega_p}^2 = \frac{Ne^2}{m_e\varepsilon_0} \tag{9.1.13}$$

At the cyclotron frequency, a solution exists to the equation of motion (9.1.1) without a forcing term, which in this case is the electric field $\mathbf{E} = 0$. Thus, at this frequency, the solution blows up if the forcing term, $E_\pm$ is not zero. This is like what happens to an LC tank circuit at resonance whose current or voltage tends to infinity when the forcing term, like the voltage or current is nonzero.

Now, one can rewrite (9.1.11) in terms of the original variables P_x, P_y, E_x, E_y, or

$$\begin{aligned} P_x = \frac{P_+ + P_-}{2} = \frac{\varepsilon_0}{2}(\chi_+E_+ + \chi_-E_-) &= \frac{\varepsilon_0}{2}[\chi_+(E_x + jE_y) + \chi_-(E_x - jE_y)] \\ &= \frac{\varepsilon_0}{2}[(\chi_+ + \chi_-)E_x + j(\chi_+ - \chi_-)E_y] \end{aligned} \tag{9.1.14}$$

$$\begin{aligned} P_y = \frac{P_+ - P_-}{2j} = \frac{\varepsilon_0}{2j}(\chi_+E_+ - \chi_-E_-) &= \frac{\varepsilon_0}{2j}[\chi_+(E_x + jE_y) - \chi_-(E_x - jE_y)] \\ &= \frac{\varepsilon_0}{2j}[(\chi_+ - \chi_-)E_x + j(\chi_+ + \chi_-)E_y] \end{aligned} \tag{9.1.15}$$

[1]This is also called the gyrofrequency.

The above relationship can be expressed using a tensor where

$$\mathbf{P} = \varepsilon_0 \overline{\boldsymbol{\chi}} \cdot \mathbf{E} \tag{9.1.16}$$

where $\mathbf{P} = [P_x, P_y]$, and $\mathbf{E} = [E_x, E_y]$. From the above, $\overline{\boldsymbol{\chi}}$ is of the form

$$\overline{\boldsymbol{\chi}} = \frac{1}{2}\begin{pmatrix} (\chi_+ + \chi_-) & j(\chi_+ - \chi_-) \\ -j(\chi_+ - \chi_-) & (\chi_+ + \chi_-) \end{pmatrix} = \begin{pmatrix} -\frac{{\omega_p}^2}{\omega^2 - \Omega^2} & -j\frac{{\omega_p}^2 \Omega}{\omega(\omega^2 - \Omega^2)} \\ j\frac{{\omega_p}^2 \Omega}{\omega(\omega^2 - \Omega^2)} & -\frac{{\omega_p}^2}{\omega^2 - \Omega^2} \end{pmatrix} \tag{9.1.17}$$

Notice that in the above, when the $\mathbf{B}$ field is turned off or $\Omega = 0$, the above resembles the solution of a collisionless, cold plasma again. For the $\mathbf{B} = 0$ case with electric field in the z direction, it will drive a motion of the electron to be in the z direction. In this case, $\mathbf{v} \times \mathbf{B}$ term is zero, and the electron motion is unaffected by the magnetic field as can be seen from the Lorentz force law or (9.1.1). Hence, it behaves like a simple collisionless plasma without a biasing magnetic field.

Consequently, for the $\mathbf{B} \neq 0$ case, the above can be generalized to 3D to give

$$\overline{\boldsymbol{\chi}} = \begin{bmatrix} \chi_0 & j\chi_1 & 0 \\ -j\chi_1 & \chi_0 & 0 \\ 0 & 0 & \chi_p \end{bmatrix} \tag{9.1.18}$$

where $\chi_p = -\omega_p^2/\omega^2$.

Using the fact that $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0(\overline{\mathbf{I}} + \overline{\boldsymbol{\chi}}) \cdot \mathbf{E} = \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}$, the above implies that

$$\overline{\boldsymbol{\varepsilon}} = \varepsilon_0 \begin{bmatrix} 1 + \chi_0 & j\chi_1 & 0 \\ -j\chi_1 & 1 + \chi_0 & 0 \\ 0 & 0 & 1 + \chi_p \end{bmatrix} \tag{9.1.19}$$

Please notice that the above tensor is a hermitian tensor. We shall learn later that this is the hallmark of a lossless medium.

Another characteristic of a gyrotropic medium is that a linearly polarized wave will rotate when passing through it. This is the Faraday rotation effect [69], which we shall learn later. This phenomenon poses a severe problem to Earth-to-satellite communication, using linearly polarized wave as it requires the alignment of the Earth-to-satellite antennas. This can be avoided using a rotatingly polarized wave, called a circularly polarized wave that we shall learn in the next section. Also, the ionosphere of the Earth is highly dependent on temperature, and the effect of the Sun. The fluctuation of particles in the ionosphere gives rise to scintillation effects due to electron motion and collision that affect radio wave communication systems [71].

9.2 Wave Polarization

Studying wave polarization is very important for communication purposes [31]. A wave whose electric field is pointing in the x direction while propagating in the z direction is a linearly polarized (LP) wave. The same can be said of one with electric field polarized in the y direction. It turns out that a linearly polarized wave suffers from Faraday rotation when it propagates through the ionosphere. For instance, an x polarized wave can become a y polarized wave due to Faraday rotation. So its polarization becomes ambiguous: to overcome this, Earth to satellite communication is done with circularly polarized (CP) waves [72]. So even if the electric field vector is rotated by Faraday's rotation, it remains to be a CP wave. We will study these polarized waves next.

We can write a general uniform plane wave propagating in the z direction as

$$\mathbf{E} = \hat{x}E_x(z,t) + \hat{y}E_y(z,t) \tag{9.2.1}$$

Clearly, $\nabla \cdot \mathbf{E} = 0$, and $E_x(z,t)$ and $E_y(z,t)$, by the principle of linear superposition, are solutions to the one-dimensional wave equation. For a time harmonic field, the two components may not be in phase, and we have in general

$$E_x(z,t) = E_1 \cos(\omega t - \beta z) \tag{9.2.2}$$

$$E_y(z,t) = E_2 \cos(\omega t - \beta z + \alpha) \tag{9.2.3}$$

where α denotes the phase difference between these two wave components. We shall study how the linear superposition of these two components behaves for different α's. First, we set $z = 0$ to observe this field. Then

$$\mathbf{E} = \hat{x}E_1 \cos(\omega t) + \hat{y}E_2 \cos(\omega t + \alpha) \tag{9.2.4}$$

For $\alpha = \frac{\pi}{2}$

$$E_x = E_1 \cos(\omega t),\ E_y = E_2 \cos(\omega t + \pi/2) \tag{9.2.5}$$

Next, we evaluate the above for different ωt's

$$\omega t = 0, \qquad E_x = E_1, \qquad E_y = 0 \tag{9.2.6}$$

$$\omega t = \pi/4, \qquad E_x = E_1/\sqrt{2}, \qquad E_y = -E_2/\sqrt{2} \tag{9.2.7}$$

$$\omega t = \pi/2, \qquad E_x = 0, \qquad E_y = -E_2 \tag{9.2.8}$$

$$\omega t = 3\pi/4, \qquad E_x = -E_1/\sqrt{2}, \qquad E_y = -E_2/\sqrt{2} \tag{9.2.9}$$

$$\omega t = \pi, \qquad E_x = -E_1, \qquad E_y = 0 \tag{9.2.10}$$

The tip of the vector field $\mathbf{E}$ traces out an ellipse as show in Figure 9.1. With the thumb pointing in the z direction, and the wave rotating in the direction of the fingers, such a wave is called left-hand elliptically polarized (LHEP) wave.

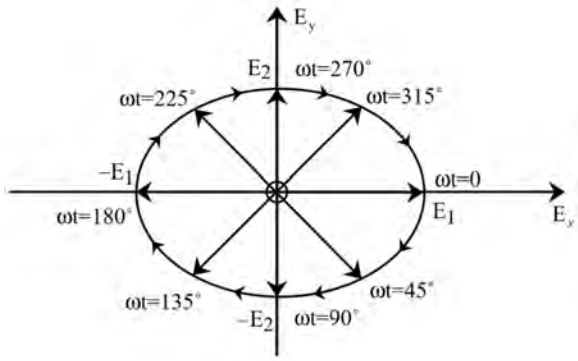

Figure 9.1: If one follows the tip of the electric field vector, it traces out an ellipse as a function of time t.

When $E_1 = E_2$, the ellipse becomes a circle, and we have a left-hand circularly polarized (LHCP) wave. When $\alpha = -\pi/2$, the wave rotates in the counter-clockwise direction, and the wave is either right-hand elliptically polarized (RHEP), or right-hand circularly polarized (RHCP) wave depending on the ratio of E_1/E_2. Figure 9.2 shows the different polarizations of the wave wave for different phase differences and amplitude ratio. Figure 9.3 shows a graphic picture of a CP wave propagating through space.

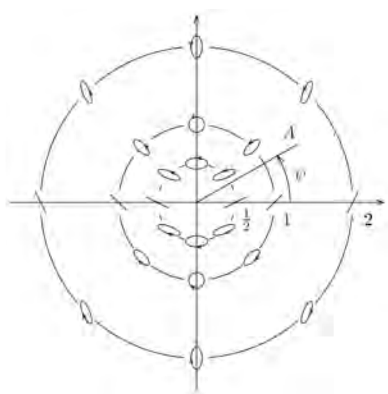

Figure 9.2: Due to different phase difference between the E_x and E_y components of the field, and their relative amplitudes E_2/E_1, different polarizations will ensure. The arrow indicates the direction of rotation of the field vector. In this figure, $\psi = -\alpha$ in our notes, and $A = E_2/E_1$ (courtesy of J.A. Kong, Electromagnetic Wave Theory [31]).

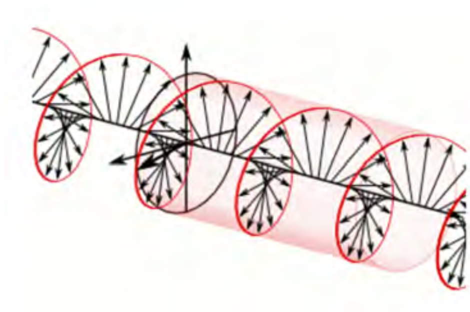

Figure 9.3: The rotation of the field vector of a right-hand circular polarization wave as it propagates in the right direction [73] (courtesy of Wikipedia).

9.2.1 Arbitrary Polarization Case and Axial Ratio

As seen before, the tip of the field vector traces out an ellipse in space as it propagates. The axial ratio (AR) is the ratio of the major axis to the minor axis of this ellipse. It is an important figure of merit for designing CP antennas (antennas that will radiate CP or circularly polarized waves). The closer is this ratio to 1, the better is the antenna design. We will discuss the general polarization and the axial ratio of a wave.

For the general case for arbitrary α, we let

$$E_x = E_1 \cos\omega t,\ E_y = E_2\cos(\omega t + \alpha) = E_2(\cos\omega t\cos\alpha - \sin\omega t\sin\alpha) \tag{9.2.11}$$

Then from the above, expressing E_y in terms of E_x, one gets

$$E_y = \frac{E_2}{E_1}E_x\cos\alpha - E_2\left[1 - \left(\frac{E_x}{E_1}\right)^2\right]^{1/2}\sin\alpha \tag{9.2.12}$$

Rearranging and squaring, we get

$$a{E_x}^2 - bE_xE_y + c{E_y}^2 = 1 \tag{9.2.13}$$

where

$$a = \frac{1}{{E_1}^2\sin^2\alpha}, \quad b = \frac{2\cos\alpha}{E_1E_2\sin^2\alpha}, \quad c = \frac{1}{{E_2}^2\sin^2\alpha} \tag{9.2.14}$$

After letting $E_x \to x$, and $E_y \to y$, equation (9.2.13) is of the form,

$$ax^2 - bxy + cy^2 = 1 \tag{9.2.15}$$

The equation of an ellipse in its self coordinates is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1 \tag{9.2.16}$$

where A and B are axes of the ellipse as shown in Figure 9.4. We can transform the above back to the (x, y) coordinates by letting

$$x' = x\cos\theta - y\sin\theta \tag{9.2.17}$$

$$y' = x\sin\theta + y\cos\theta \tag{9.2.18}$$

to get

$$x^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - xy\sin 2\theta\left(\frac{1}{A^2} - \frac{1}{B^2}\right) + y^2\left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2}\right) = 1 \tag{9.2.19}$$

Comparing (9.2.13) and (9.2.19), one gets

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\cos\alpha E_1 E_2}{{E_2}^2 - {E_1}^2}\right) \tag{9.2.20}$$

$$\mathrm{AR} = \left(\frac{1+\Delta}{1-\Delta}\right)^{1/2} > 1 \tag{9.2.21}$$

where AR is the axial ratio and

$$\Delta = \left(1 - \frac{4{E_1}^2{E_2}^2\sin^2\alpha}{\left({E_1}^2 + {E_2}^2\right)^2}\right)^{1/2} \tag{9.2.22}$$

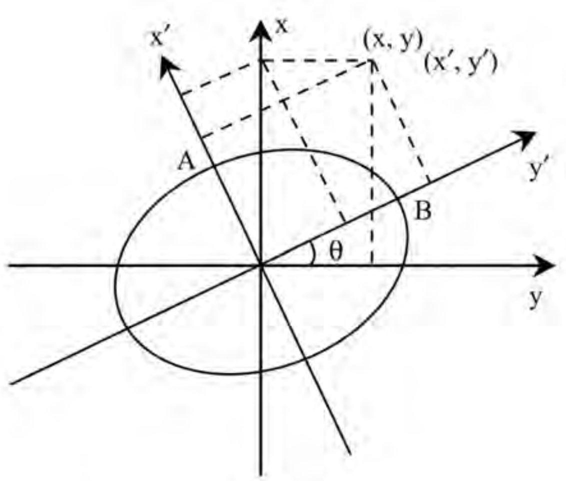

Figure 9.4: This figure shows the parameters used to derive the axial ratio (AR) of an elliptically polarized wave.

9.3 Polarization and Power Flow

For a linearly polarized wave,

$$\mathbf{E} = \hat{x}E_0 \cos(\omega t - \beta z), \quad \mathbf{H} = \hat{y}\frac{E_0}{\eta}\cos(\omega t - \beta z) \tag{9.3.1}$$

Hence, the instantaneous power becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{{E_0}^2}{\eta}\cos^2(\omega t - \beta z) \tag{9.3.2}$$

indicating that for a linearly polarized wave, the instantaneous power is function of both time and space. It travels as lumps of energy through space. In the above E_0 is the amplitude of the linearly polarized wave.

Next, we look at power flow for for elliptically and circularly polarized waves. It is to be noted that in the phasor world or frequency domain, (9.2.1) becomes

$$\mathbf{E}(z,\omega) = \hat{x}E_1 e^{-j\beta z} + \hat{y}E_2 e^{-j\beta z + j\alpha} \tag{9.3.3}$$

For LHEP,

$$\mathbf{E}(z,\omega) = e^{-j\beta z}(\hat{x}E_1 + j\hat{y}E_2) \tag{9.3.4}$$

whereas for LHCP

$$\mathbf{E}(z,\omega) = e^{-j\beta z}E_1(\hat{x} + j\hat{y}) \tag{9.3.5}$$

For RHEP, the above becomes

$$\mathbf{E}(z,\omega) = e^{-j\beta z}(\hat{x}E_1 - j\hat{y}E_2) \tag{9.3.6}$$

whereas for RHCP, it is

$$\mathbf{E}(z,\omega) = e^{-j\beta z}E_1(\hat{x} - j\hat{y}) \tag{9.3.7}$$

Focussing on the circularly polarized wave,

$$\mathbf{E} = (\hat{x} \pm j\hat{y})E_0 e^{-j\beta z} \tag{9.3.8}$$

Using that

$$\mathbf{H} = \frac{\boldsymbol{\beta} \times \mathbf{E}}{\omega\mu},$$

where $\boldsymbol{\beta} = \hat{z}\beta$, then

$$\mathbf{H} = (\mp\hat{x} - j\hat{y})j\frac{E_0}{\eta}e^{-j\beta z} \tag{9.3.9}$$

where $\eta = \sqrt{\mu/\varepsilon}$. Therefore,

$$\mathbf{E}(t) = \hat{x}E_0 \cos(\omega t - \beta z) \pm \hat{y}E_0 \sin(\omega t - \beta z) \tag{9.3.10}$$

$$\mathbf{H}(t) = \mp\hat{x}\frac{E_0}{\eta} \sin(\omega t - \beta z) + \hat{y}\frac{E_0}{\eta} \cos(\omega t - \beta z) \tag{9.3.11}$$

Then the instantaneous power becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{{E_0}^2}{\eta} \cos^2(\omega t - \beta z) + \hat{z}\frac{{E_0}^2}{\eta} \sin^2(\omega t - \beta z) = \hat{z}\frac{{E_0}^2}{\eta} \tag{9.3.12}$$

In other words, a CP wave delivers constant power independent of space and time.

It is to be noted that the complex Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* \tag{9.3.13}$$

are real both for linearly, circularly, and elliptically polarized waves. This is because there is no reactive power in a plane wave of any polarization: the stored energy in the plane wave cannot be returned to the source!